Vertical Differentiation through Product Design*

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Abstract

I study pricing and product design choices of multiproduct firms in a model of directed search. Product design introduces vertical differentiation à la Gabszewicz and Thisse (1979) as well as Shaked and Sutton (1982). While all consumers have a preference for a more niche product design, consumers with lower search costs benefit relatively more. Firms gain from dispersion in tastes through product design and choose maximum differentiation in equilibrium. The firm with the broader product design sets a lower price and attracts consumers with high search costs.

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1 Introduction

Consider a retailer which determines its product assortment. As it strives to establish a consistent brand reputation, the company faces a strategic decision on product design. It could either offer polarizing products ("niche products"), that are hated by some consumers and loved by others, or generic products ("broad products"), that serve most consumers relatively well.

There are two economic interpretations of how product design affects the heterogeneity of consumers' tastes. A literal interpretation pertains to the determination of a product's characteristics. On the one hand, a fashion retailer could focus on mainstream fashion, characterized by classic designs, standard sizes and neutral colors, that fits into each and every one's wardrobe. On the other hand, it could offer more polarizing fashion pieces, potentially at higher prices, that have an extravagant style and are tailored to very specific needs. For instance, Urban Outfitters could be seen as a fashion retailer with a niche product design. It describes its brand as "a lifestyle retailer dedicated to inspiring customers through a unique combination of product, creativity and cultural understanding."[1] The German newspaper "stern" states that "an Urban Outfitters store resembles the apartment of a chaotic friend with a penchant for collecting and a love for kitsch." (Pientka, 2008) In contrast, H&M seems to take a more generic approach on product design, describing itself as a "fashion brand, offering the latest styles and inspiration for all — always." It also explicitly mentions to offer "affordable wardrobe essentials" and stresses its focus on low prices.[2]

An alternative interpretation of product design revolves around information provision.[3] Suppose that consumers only receive a noisy signal about their fit with a specific product. A retailer can influence the accuracy of this signal by providing information about its products' attributes. Precise information allows consumers to carefully assess a product, thereby increasing heterogeneity in tastes. Extensive information provision could thus resemble a niche product design. For example, there are specialized stores for many consumables such as tea, coffee or cheese. While these charge higher prices than supermarkets, they allow consumers to smell and even taste their products, enabling them to make a more informed choice.

In this paper, I provide a rationale for why both – niche and broad product designs – as well as the associated differences in pricing can be observed in practical settings. I study

[1]Source: https://www.urbn.com/our-brands/urban-outfitters, last accessed November 22, 2023.
[2]Source: https://hmgroup.com/brands/hm/, last accessed November 22, 2023.
[3]This interpretation was brought up by Johnson and Myatt (2006).

the product design and pricing choices of multiproduct firms in a model of directed search, where consumers observe product design and aggregate pricing. Product design assumes a different role than in models of random search (Bar-Isaac et al., 2012, 2023; Larson, 2013). Under random search, firms set product design to optimally exploit a given set of consumers that visit their products. Under directed search, product design choices also influence the share of consumers that visit a firm's products.

My analysis is based on a duopoly model, where firms both offer a continuum of products and engage in a multi-stage game. First, they simultaneously set product design. Second, they compete in prices. Consumers, which face heterogeneous search cost, then observe the product design choices and average prices of both firms and can sequentially search through their products.

At equal prices, consumers exhibit a preference for a niche product design. As the expected gains from search are driven by the chance to obtain a high match value, consumers benefit from a higher dispersion of match values through a more polarized product design. However, consumers with a higher search cost gain relatively less because the expected number of products searched under the optimal stopping rule of Weitzman (1979) is higher for a more niche product design. Product design thus introduces vertical differentiation à la Gabszewicz and Thisse (1979) as well as Shaked and Sutton (1982). While all consumers prefer a more polarized product design, the intensity of preferences is based on their respective search cost.

Under asymmetric product design, the firm with the more polarized product design sets a higher price to exploit its competitive advantage in the second-stage pricing equilibrium in which both firms set the same price for each of their products (symmetric pricing). Consumers select into products according to their search cost. Consumers whose search cost are sufficiently low (high) inspect only products of the firm with the more niche (broader) product design. Under a parameter restriction, a unique (pure-strategy) pricing equilibrium is obtained for each subgame. Depending on product design choices, either both firms make sales in an interior equilibrium or the firm with the more polarized product design captures the entire market in a corner equilibrium. If both firms offer the same product design, Bertrand competition is obtained and both firms make zero profits.

Introducing differentiation through product design can increase the profits of both firms. If the range of product designs is such that no firm can unilaterally enforce a corner pricing equilibrium by setting the most niche product design, firms choose maximum differentiation in equilibrium: While one firm sets the broadest product design, the other firm sets the most niche product design. If firms can unilaterally enforce a corner pricing equilibrium, there is a continuum of equilibria involving at least one firm setting the most niche design. Special cases are constituted by the equilibrium in which both firms set the most niche design, which results in Betrand competition at the pricing stage and maximizes consumer surplus, and the equilibria with maximum differentiation in product design, which give rise to the most profitable corner pricing equilibria.

1.1 Plan of the paper

The following subsection provides an overview of the relevant literature. In Section 2, I present the model, which is solved backwards in the subsequent sections: Section 3 characterizes consumer search, Section 4 pricing and Section 5 product design. I conclude in Section 6. Most proofs are collected in the Appendix.

1.2 Related literature

This paper builds upon the literature on consumer search for differentiated products and is in particular related to the literature on product design in search markets. Models of consumer search can be categorized into three groups. Classical models such as Wolinsky (1986), Anderson and Renault (1999) and Moraga-González et al. (2017) constitute random search models. Consumers search through ex-ante identical products in random order. In contrast, models of ordered search (Armstrong et al., 2009; Zhou, 2011) assume that consumers are obligated to follow an exogenously determined search order.[4] Models of directed search (Choi et al., 2018; Haan et al., 2018) endogenize search order. Consumers determine which product to search first based on observable characteristics such as price.

Most of the work on product design in search markets relies on random search models.[5] Bar-Isaac et al. (2012) study product design choices of vertically differentiated single-product firms. They model changes in product design as demand rotations à la Johnson and Myatt (2006). Using their result on a monopolist's preference for extremes, Bar-Isaac et al. (2012) show that low-quality firms choose the most niche design and high-quality firms choose the broadest design. Larson (2013) studies a similar model without vertical differentiation, which employs a special form of demand rotations. Depending on the search cost, equilibrium design choices entail only the most niche design, only the broadest design or

[4] Arbatskaya (2007) examines ordered price search with homogeneous products.

[5] Choi (2021), who studies optimal pricing and product design choices under ordered search, constitutes an exception.

both extreme designs. Bar-Isaac et al. (2023) propose another framework for product design that allows for the optimality of intermediate design choices and characterize the optimal design choice in search markets.

The comparative statics with respect to search costs are common to the three studies. A decrease in (homogeneous) search costs leads to (weakly) more polarizing product design choices in equilibrium. Intuitively, lower search costs lead to "pickier" consumers, who buy only at high match values. Niche products offer an increased likelihood for exceptionally high match values and therefore become more attractive in the presence of low search costs. As shown by Bar-Isaac et al. (2012), this finding constitutes a potential explanation for the empirical observation of the "long tail effect" (Anderson, 2004; Brynjolfsson et al., 2011), which describes the increasing importance of fringe products for sales in digital markets.[6]

I contribute to the literature by studying product design choices of multiproduct firms in a directed search model.[7] It turns out that my work is akin to models of vertical differentiation, which were pioneered by Gabszewicz and Thisse (1979) as well as Shaked and Sutton (1982). In these models, consumers exhibit heterogeneous tastes for quality. Vertical differentiation has a similar impact to horizontal differentiation, leading to dispersed utilities among consumers and enabling firms to charge higher prices than for homogeneous products.[8] If firms choose product quality before competing in prices, they do not all choose the highest quality level – even if this comes at zero cost. I endogenize such a model of quality choice in a consumer search framework. Product design resembles quality, as all consumers prefer to visit the firm with the most niche design when prices are equal. However, the desirability of a more niche design decreases with search cost. Consequently, search cost heterogeneity gives rise to heterogeneous tastes for product design.

Consumers thus sort into products with different designs based on their respective search costs. A similar mechanism is apparent in Salop (1977), which examines a monopolist selling a homogeneous good through multiple outlets. Consumers face heterogeneous search costs when sequentially searching for the price of these outlets. Salop (1977) shows that

[6] Yang (2013) provides another framework to theoretically study the long-tail effect. He assumes that products can be assigned to product categories that are demanded by varying numbers of consumers.

[7] To the best of my knowledge, Song (2017) constitutes the only paper that studies directed search based on product design. However, the focus of this paper lies on which search orders can be sustained in equilibrium. Asymmetric choices of product design are given exogeneously. Homogeneous search costs, single-product firms and the non-observability of prices constitute other important differences from my model.

[8] Cremer and Thisse (1991) even show that a large class of Hotelling-type models can be written as a special case of a vertical differentiation model. However, corresponding vertical differentiation models require marginal costs that are increasing in quality. A similar result is thus not obtained for the representation of vertical differentiation models as models of horizontal differentiation.

firms can benefit from setting different prices across outlets when there is a positive correlation between consumers' search cost and demand. Price dispersion then serves as a price discrimination device against consumers with higher search cost, which – on average – visit less outlets and therefore buy at higher prices.

2 Setting

There are two multiproduct firms, which both offer a continuum of products, and a continuum of risk-neutral consumers of mass 1. Consumers have unit demand and the value of their outside option is normalized to zero. Each consumer j's tastes are described by a conditional utility function (net of any search costs) of the form:

$$u_{jki}(p_{ki}) = \epsilon_{jki} - p_{ki}$$

if buying product i from firm k at price p_{ki}. The stochastic match value between consumer j and firm k's product i is denoted by ϵ_{jki}. Match values and prices of specific products are ex-ante unknown to consumers and can be discovered through sequential search. While consumers can decide whether to search an additional product of firm 1 or of firm 2 (directed search), they search through the firms' products at random.[9] Search costs are heterogeneous.[10] Consumer j's search cost s_j is drawn independently from a differentiable cumulative distribution function (CDF) G with support $[\underline{s}, \bar{s}]$. I assume that the corresponding density g is strictly positive and that G and $1-G$ are both log-concave.[11]

Both firms produce a continuum of products at zero marginal cost. They choose prices and product design for their products. Product design affects the distribution of match values. Specifically, I follow Larson (2013) and assume that match values take the following form:

$$\epsilon_{jki} = \epsilon_\mu + \alpha_k \cdot z_{jki}$$

The baseline utility for the good is represented by the constant $\epsilon_\mu > 0$. I assume that ϵ_μ is sufficiently high such that no consumer would end up taking the outside option. The second term $\alpha_k \cdot z_{jki}$ captures idiosyncratic consumer preferences for specific products. The realization of z_{jki} is assumed to be drawn independently across consumers, firms and products

[9] Since all products of a given firm are ex-ante identical, consumers cannot do better than under random ordering.

[10] Among others, De los Santos (2018) provides empirical evidence for search cost heterogeneity.

[11] See Bagnoli and Bergstrom (2005) for a technical discussion of log-concavity.

from a CDF F, which is assumed to be continuously differentiable, to have a strictly positive density f and to represent a random variable of mean zero. Its support is $[\underline{z}, \bar{z}]$, where $\bar{z} > 0 > \underline{z}$. Product design is represented by parameter α_k, which is chosen by firms from an interval $[\underline{\alpha}, \bar{\alpha}]$, where $\bar{\alpha} > \underline{\alpha} > 0$. There is no cost associated with product design.

A firm's product design choice affects the dispersion of consumers' match values. The random component of the match value is scaled by α_k. A higher α_k, which represents a more polarizing design, would lead to more extreme match value realizations. In the spirit of Anderson and Renault (1999),[12] product design choices thus affect the degree of horizontal differentiation between products.[13]

As argued in the Introduction, there are two economic interpretations of α_k. On the one hand, a higher α_k could correspond to a more extravagant design of a firm's products. Their attributes, such as style, color, and fit in the case of fashion items, can be selected to influence their polarization among consumers. On the other hand, a higher α_k could be resemble a more extensive information policy. In this context, α_k could be interpreted as the (continuous) number of product attributes for which a company offers precise information.

Crucially, I assume that the product design choice and the average price of each firm are observable to consumers. While consumers have to search for the individual characteristics and price of each product, they obtain information on the firm level. This captures the idea that firms build a brand reputation, which allows consumers to infer their product design and average price. For instance, some clothing brands are known for offering standard products at affordable rates, while others sell eccentric fashion at exorbitant prices.

The timing of the game is as follows:

1. Firms simultaneously and publicly choose product design $\alpha_k \in [\underline{\alpha}, \bar{\alpha}]$.
2. Firms simultaneously set (non-negative) prices p_{ki} for their products.
3. Consumers observe the average prices of both firms and engage in sequential search.

At the third stage of the game, consumers need to form beliefs about the price distributions set by firms. I assume passive beliefs, which means that consumers do not revise their beliefs based on the prices they come across. Henceforth, when referring to "equilibrium," it denotes a Perfect Bayesian Equilibrium with passive beliefs.

[12] Anderson and Renault (1999) use a scaling parameter that affects the match value distribution of all products. Here, this scaling parameter is firm-specific.

[13] For an in-depth motivation and discussion of this approach to model product design, see Larson (2013).

In section 4, I show that in equilibrium each firm sets the same price for each of its products at the second stage of the game. In the following section, I solve for the optimal consumer search strategy "on-path" when consumers believe in symmetric pricing. Consumer search under more general beliefs is discussed in the Appendix. Furthermore, I assume that the gains of match search are high enough for each product design such that no consumer would always buy the first product she visited. This is formalized by the following assumption:[14]

$$\underline{\alpha} \cdot \int_{\underline{z}}^{\bar{z}} (z - \underline{z})\, dF(z) \geq \bar{s}$$

3 Consumer search

Consider a consumer with search cost $s \in [\underline{s}, \bar{s}]$. In principle, she could search products of both firms in any order. However, it turns out that it is optimal to exclusively search for products from a single firm. This follows from the optimal search rule of Weitzman (1979), which prescribes that consumers should search through products in the order determined by their reservation prices. Since products from a given firm are ex-ante identical, their reservation prices are likewise identical. If one of the firms offers products with a higher reservation price, the consumer should therefore search through all of its products first. As the firm offers a continuum of products, the consumer never visits the other firm. The optimal rule of search can be characterized in two steps. First, the optimal within-firm search is characterized. Second, reservation prices are compared across firms.

Suppose that the consumer only searches products of firm k which sets product design α_k and price p_k for all of its products. The benefit of search consists of the chance of getting a higher draw from distribution F and thus a higher match value. As shown by Kohn and Shavell (1974), optimal search is determined by a threshold value. I define this threshold value relative to the match value distribution F and call it the reservation draw r_k.[15] It is determined by the following equation:

$$\alpha_k \cdot \int_{r_k}^{\bar{z}} (z - r_k)\, dF(z) = s \tag{1}$$

At the reservation draw, the benefits of an additional search through a potential higher

[14] I will later assume that z_{jki} is uniformly distributed on $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Then, this assumption corresponds to $\bar{s} \leq \frac{1}{2} \cdot \underline{\alpha}$.

[15] The reservation price of a product is a strictly increasing linear transformation of its reservation draw.

match value equal the search costs. Whenever the consumer obtains a higher realization from F than the reservation draw, she should stop searching and buy. Otherwise, she should continue to search.

The reservation draw depends on s and α_k. Using the implicit function theorem on (1), the following comparative statics can be obtained:

$$\frac{\partial r_k}{\partial \alpha_k} = \frac{s}{\alpha_k^2 \cdot (1 - F(r_k))} > 0 \qquad \frac{\partial r_k}{\partial s} = -\frac{1}{\alpha_k \cdot (1 - F(r_k))} < 0$$

The reservation draw is decreasing in s and increasing in α_k. Consumers are "pickier" when facing a lower search cost and when the firm offers a more niche product design. The latter implies that consumers, on average, search more products as α_k increases.

Given the optimal within-firm search rule, the expected consumer surplus CS_k of searching products from firm k can be calculated. Using (1), this corresponds to the reservation price of each of the firm's products:

$$CS_k = \underbrace{\epsilon_\mu + \alpha_k \cdot \left(r_k + \frac{\int_{r_k}^{\bar{z}} (z - r_k)\, dF(z)}{1 - F(r_k)} \right)}_{\text{expected match value}} - \underbrace{\frac{s}{1 - F(r_k)}}_{\substack{\text{expected total} \\ \text{search cost}}} - p_k = \underbrace{\epsilon_\mu + \alpha_k \cdot r_k - p_k}_{\text{reservation price}}$$

The consumer ranks firms according to this measure and visits only the firm with the higher surplus. She keeps searching until she discovers a good enough match. The demand of a firm is equal to the share of consumers that visit that firm.

The comparative statics of consumer surplus with respect to α_k are as follows:

$$\frac{\partial CS_k}{\partial \alpha_k} = r_k + \alpha_k \cdot \frac{\partial r_k}{\partial \alpha_k} = r_k + \frac{\int_{r_k}^{\bar{z}} (z - r_k)\, dF(z)}{1 - F(r_k)} = \frac{\int_{r_k}^{\bar{z}} z\, dF(z)}{1 - F(r_k)} > 0$$

where the second equality follows from (1) and the inequality arises due to the fact that F has mean zero. At equal prices, consumers have a preference for products with higher α_k. This is not surprising, as the benefit of search is driven by the chance of obtaining a high match value. Consumers gain from an increase in the dispersion of match values. Product design introduces a sort of vertical differentiation, wherein a niche product design resembles a product of higher quality.

While consumers prefer more niche products regardless of search costs, their intensity of preferences varies with search costs:

$$\frac{\partial^2 CS_k}{\partial \alpha_k \partial s} = -\frac{f(r_k)\cdot s}{\alpha_k^2 \cdot (1-F(r_k))^3} < 0 \tag{2}$$

Consumers with a lower search cost benefit relatively more from more polarizing product designs. This is intuitive due to the comparative statics of the reservation draw with respect to product design. As α_k increases, consumers become "pickier" and search more products on average. An increased duration of search is less harmful for consumers with a lower search cost.

4 Pricing

As stated before, firms set the same price for each of their products in equilibrium.

Lemma 1. *Any (pure-strategy) equilibrium involves symmetric pricing of both firms.*

Proof: See Appendix.

There is a simple intuition behind this result: Regardless of their beliefs, which are only affected by changes in the average price, consumers would be more likely to buy products with lower price realizations when a firm sets a non-degenerate price distribution. It would thus always be optimal for this firm to deviate to price distributions that eliminate some of the dispersion in prices while keeping the mean price constant and obtain a higher average purchase price.

The same reasoning establishes that symmetric pricing is optimal for any symmetric pricing equilibrium candidate. While the number of consumers who visit a firm (and eventually buy one of its products) solely depends on the observable average price, any pricing scheme involving price dispersion would lead to a lower average purchase price. Following this spirit, I assume that consumers also believe in symmetric pricing when observing an off-path mean price. It then suffices to check for deviations to non-degenerate price distributions, as these capture the most profitable deviation for any off-path mean price.

My model is then formally equivalent to a (single-product) duopoly model of quality choice, in which firms set α_k and a single price p_k and consumers have a taste parameter s_j and get utility CS_k when choosing firm k, which depends on α_k and s_j. Despite that, the analysis of the pricing subgame does not directly follow from classical models of vertical differentiation, since they rely on a very specific assumption on the functional form of utility.[16] In order to obtain tractable conditions for equilibrium existence and uniqueness, I from now on assume that F follows a uniform distribution with support $\left[-\frac{1}{2},\frac{1}{2}\right]$. Using (1), explicit expressions for the reservation draw and the expected consumer surplus can be derived as functions of the search cost:

$$r_k(s) = \frac{1}{2} - \sqrt{2\cdot\frac{s}{\alpha_k}} \qquad CS_k(s) = \epsilon_\mu + \frac{\alpha_k}{2} - \sqrt{2s\alpha_k} - p_k$$

I define the net consumer surplus (NCS) to capture the consumer surplus net of price and baseline utility:[17]

$$NCS_k(s) = \frac{\alpha_k}{2} - \sqrt{2s\alpha_k}$$

If firms set the same product design (α_1=α_2), products are ex-ante homogeneous. Firms engage in Bertrand competition and price at marginal cost. In the following, I solve for the pricing equilibrium with asymmetric product design. Without loss of generality, I assume that $\alpha_1 > \alpha_2$. An immediate Lemma characterizing demand follows from the discussion in Section 3.

Lemma 2. *There exists a unique $\hat{s} \in [\underline{s}, \bar{s}]$ such that consumers visit*

- *firm 1 if $s < \hat{s}$ and*
- *firm 2 if $s > \hat{s}$.*

It holds that $\hat{s} = \bar{s}$ if $NCS_1(\bar{s}) - NCS_2(\bar{s}) \geq p_1 - p_2$ and $\hat{s} = \underline{s}$ if $NCS_1(\underline{s}) - NCS_2(\underline{s}) \leq p_1 - p_2$. Otherwise, $\hat{s} \in (\underline{s}, \bar{s})$ and solves $NCS_1(\hat{s}) - NCS_2(\hat{s}) = p_1 - p_2$.

Proof: Follows from the fact that $NCS_1(s) - NCS_2(s)$ is continuous and strictly decreasing in s, which is implied by (2).

The heterogeneous taste for product design leads to a selection of consumers according to their search costs. If $\hat{s}$ solves $NCS_1(\hat{s}) - NCS_2(\hat{s}) = p_1 - p_2$, there is a threshold search cost at which a consumer is exactly indifferent between visiting firm 1 and firm 2. Consumers with a lower (higher) search cost have a stronger (weaker) preference for a niche design and buy at firm 1 (2). Using Lemma 2, the profit functions of both firms can be derived:

$$\pi_1(p_1,p_2) = p_1 \cdot G\big(\hat{s}(p_1,p_2)\big) \qquad \pi_2(p_1,p_2) = p_2 \cdot \Big(1 - G\big(\hat{s}(p_1,p_2)\big)\Big)$$

[16]In the Appendix, I analyze a textbook model of vertical differentiation under the same technical assumptions as in my model.

[17]This is stricly increasing in α_k due to the assumption that $\bar{s} \leq \frac{1}{2}\cdot\underline{\alpha}$.

The following Lemma shows that it is optimal for firms to set prices such that consumers are indeed indifferent at the threshold search cost:

Lemma 3. *Given the price of firm 2 (1), firm 1 (2) is weakly better off to set a price such that*

$$NCS_1(\hat{s}) - NCS_2(\hat{s}) = p_1 - p_2 \tag{3}$$

Proof: See Appendix.

Under this restriction, there is a one-to-one mapping between prices that could be optimal for a firm and the resulting threshold search costs given the price of the other firm. The pricing problem of each firm can be rewritten into an equivalent formulation,in which firms choose demand via determining the threshold search cost:[18]

$$\text{Firm 1: } \max_{\hat{s} \in [\underline{s}, \bar{s}]} \left(p_2 + NCS_1(\hat{s}) - NCS_2(\hat{s})\right) \cdot G(\hat{s})$$

$$\text{Firm 2: } \max_{\hat{s} \in [\underline{s}, \bar{s}]} \left(p_1 - NCS_1(\hat{s}) + NCS_2(\hat{s})\right) \cdot \left(1 - G(\hat{s})\right)$$

In equilibrium, the threshold search cost could either lie in the interior of the support of G or at $\bar{s}$, indicating that all consumers buy from firm 1. There cannot be an equilibrium with $\hat{s} = \underline{s}$. This would imply that firm 1 does not make any sales and can profitably deviate to $p_1 = p_2 + NCS_1(\bar{s}) - NCS_2(\bar{s})$, get the entire demand and make positive profits.

I first focus on interior equilibria. The first-order conditions can be solved for the equilibrium prices depending on the implicitly defined threshold search cost $\hat{s}^*$:

$$p_1^* = (\sqrt{\alpha_1} - \sqrt{\alpha_2}) \cdot \frac{1}{\sqrt{2\hat{s}^*}} \cdot \frac{G(\hat{s}^*)}{g(\hat{s}^*)} \qquad p_2^* = (\sqrt{\alpha_1} - \sqrt{\alpha_2}) \cdot \frac{1}{\sqrt{2\hat{s}^*}} \cdot \frac{1 - G(\hat{s}^*)}{g(\hat{s}^*)}$$

Using these equilibrium prices, the necessary condition (3) takes the following form:

$$\underbrace{(\sqrt{\alpha_1} - \sqrt{\alpha_2}) \cdot \frac{1}{\sqrt{2\hat{s}^*}} \cdot \frac{2G(\hat{s}^*) - 1}{g(\hat{s}^*)}}_{\Delta p} = \underbrace{(\alpha_1 - \alpha_2) \cdot \frac{1}{2} - (\sqrt{\alpha_1} - \sqrt{\alpha_2}) \cdot \sqrt{2\hat{s}^*}}_{NCS_1(\hat{s}^*) - NCS_2(\hat{s}^*)} \tag{4}$$

While the right-hand side (RHS) of this equation is strictly positive as $\alpha_1 > \alpha_2$, the left-

[18] Strictly speaking, a restriction of the support of firm 2's action set would be needed. Given a low p_1, setting low threshold search costs can involve setting a negative price.

hand side (LHS) is strictly negative whenever $\hat{s}^*$ is smaller than the median of G, which I denote med(G). Any equilibrium requires therefore that $\hat{s}^* \geq \text{med}(G)$. Condition (4) can be simplified to:

$$\frac{2G(\hat{s}^*)-1}{g(\hat{s}^*)} = (\sqrt{\alpha_1}+\sqrt{\alpha_2})\cdot\frac{\sqrt{\hat{s}^*}}{\sqrt{2}} - 2\hat{s}^* \tag{5}$$

The LHS of this equation is weakly increasing in s due to the log-concavity of G and $1-G$. If the RHS is strictly decreasing in s for $s \geq \text{med}(G)$, there is at most one candidate for an interior equilibrium. The slope of the RHS equals $(\sqrt{\alpha_1}+\sqrt{\alpha_2})\cdot\frac{1}{2^{1.5}\cdot\sqrt{s}} - 2$. A sufficient condition for the uniqueness of the interior equilibrium candidate in any subgame is given by $\bar{\alpha} < 8\cdot\text{med(G)}$.

With this condition, the existence of an interior equilibrium candidate is determined by a simple criterion. There is an interior equilibrium candidate if and only if the LHS of equation (5) exceeds its RHS when both are evaluated at $\bar{s}$. This holds true under the following condition:

$$\sqrt{\alpha_1}+\sqrt{\alpha_2} < \frac{\sqrt{2}}{g(\bar{s})\cdot\sqrt{\bar{s}}} + 2\cdot\sqrt{2}\cdot\sqrt{\bar{s}} \tag{6}$$

It turns out that the negation of (6) constitutes a necessary condition for a corner equilibrium. A corner equilibrium requires that $p_2 = 0$. Otherwise, either firm 1 could set a higher price and obtain the entire demand or firm 2 could set a lower price and make positive profits. Firm 1 solves the following optimization problem:

$$\max_{\hat{s}\in[\underline{s},\bar{s}]} \big(NCS_1(\hat{s}) - NCS_2(\hat{s})\big)\cdot G(\hat{s}) \tag{7}$$

A corner equilibrium is obtained if the objective function is maximized at $\bar{s}$. A necessary condition is local optimality. The derivative of the objective function evaluated at $\bar{s}$ must be weakly positive. This holds true if and only if (6) is violated.

Instead of a condition on $\sqrt{\alpha_1}+\sqrt{\alpha_2}$, one might have anticipated that a corner equilibrium would occur if product design choices are sufficiently dispersed. There are however two effects of changes in product design. First, a higher α_1 and a lower α_2 increase the competitive advantage of firm 1 and thus its demand at given prices. Second, a higher α_1 and a lower α_2 increase the degree of differentiation. This makes it more attractive for firm 1 to exploit a smaller subset of consumers with a low search cost instead of serving the entire market. While for changes in α_1 the first effect dominates the second, the reverse holds true for changes in α_2.

Under the preceding assumptions, interior and corner equilibria are mutually exclusive and there is exactly one equilibrium candidate for each subgame. Using additional sufficient conditions, which are derived in the Appendix, the existence of a unique (pure-strategy) equilibrium can be proven. The following Proposition summarizes the results of the pricing stage when $\alpha_1 > \alpha_2$.

Proposition 1. *Suppose that* $\sqrt{\bar{\alpha}} < 2 \cdot \sqrt{2\underline{s}} - \frac{\sqrt{2}}{med(G)} \cdot \frac{1}{4 \cdot g(med(G))}$. *Then, there exists a unique pure-strategy equilibrium at the pricing stage that can be characterized as follows:*

- *If (6) holds, there is an interior equilibrium:* $\hat{s}^* \in [med(G), \bar{s}]$ *is implicitly defined by (5),* $p_1^* = (\sqrt{\alpha_1} - \sqrt{\alpha_2}) \cdot \frac{1}{\sqrt{2\hat{s}^*}} \cdot \frac{G(\hat{s}^*)}{g(\hat{s}^*)}$, $p_2^* = (\sqrt{\alpha_1} - \sqrt{\alpha_2}) \cdot \frac{1}{\sqrt{2\hat{s}^*}} \cdot \frac{1-G(\hat{s}^*)}{g(\hat{s}^*)}$.
- *If (6) does not hold, there is a corner equilibrium:* $\hat{s}^* = \bar{s}$, $p_1^* = NCS_1(\bar{s}) - NCS_2(\bar{s})$, $p_2^* = 0$.

Proof: See Appendix.

5 Product Design

Section 4 determines payoffs for each pair $(\alpha_1, \alpha_2) \in [\underline{\alpha}, \bar{\alpha}]^2$ under the previously introduced assumptions that $\bar{s} \leq \frac{1}{2} \cdot \underline{\alpha}$ and $\sqrt{\bar{\alpha}} < 2 \cdot \sqrt{2\underline{s}} - \frac{\sqrt{2}}{\text{med}(G)} \cdot \frac{1}{4 \cdot g(\text{med}(G))}$. Comparative statics of the equilibrium profits in an interior pricing equilibrium yield the following Lemma.[19]

Lemma 4. *While firm 1's interior equilibrium profits are strictly increasing in* α_1, *firm 2's interior equilibrium profits are strictly decreasing in* α_2. *Thus, any equilibrium that gives rise to an interior pricing equilibrium requires that one firm sets* $\bar{\alpha}$ *and the other firm sets* $\underline{\alpha}$.

Proof: See Appendix.

This demonstrates that implementing vertical differentiation via product design constitutes a source of market power. Despite firm 2 facing a competitive disadvantage when adopting a broad product design, it benefits from doing so due to reduced competition resulting from more dispersed preferences of consumers.

It follows from Lemma 4, that there is a unique pair of equilibria under a restriction on the support of product designs $[\underline{\alpha}, \bar{\alpha}]$. Suppose that condition (6) holds when evaluated at $\bar{\alpha}$

[19]During this section, I maintain the notation that $\alpha_1 > \alpha_2$. Firm 1 (2) generally refers to the firm with the more polarized (broader) product design in equilibrium.

and $\underline{\alpha}$:

$$\sqrt{\bar{\alpha}} + \sqrt{\underline{\alpha}} < \frac{\sqrt{2}}{g(\bar{s}) \cdot \sqrt{\bar{s}}} + 2 \cdot \sqrt{2} \cdot \sqrt{\bar{s}} \tag{8}$$

This restriction ensures that no firm can unilaterally enforce a corner pricing equilibrium by setting the most niche product design $\bar{\alpha}$. Then, there cannot be an equilibrium that does not give rise to an interior pricing equilibrium. Otherwise, one firm would earn zero profits and could profitably deviate by setting $\underline{\alpha}$. By Lemma 4, in equilibrium, one firm opts for the most niche product design while the other firm selects the broadest product design.

If condition (8) is violated, any equilibrium still requires that one firm opts for the most niche product design. Otherwise, the firm with the more niche design could increase its profits by deviating to an even more niche design.

Lemma 5. *Suppose that condition (8) is violated. Then, in any equilibrium, at least one firm sets the most niche product design $\bar{\alpha}$.*

Proof: See Appendix.

Under condition (8), any product design $\alpha \in [\underline{\alpha}, \bar{\alpha}]$ yields zero profits against $\bar{\alpha}$ and is thus a best response. An equilibrium requires however that $\bar{\alpha}$ also constitutes a best response to α_2^*. In principle, it could be a profitable deviation for firm 1 to set $\underline{\alpha}$. Its comparative disadvantage through a broader product design could be offset by its gains due to increased dispersion in consumers' tastes. I derive conditions under which this deviation is unprofitable. In any case, this holds true for sufficiently high and sufficiently low values of α_2^*. If α_2^* lies above some threshold ($\tilde{\alpha}$), deviating to $\underline{\alpha}$ would lead to a corner pricing equilibrium and zero profits for firm 1. If α_2^* is sufficiently small, deviating to the broadest product design would yield a pricing outcome close to Bertrand competition. There is thus a continuum of equilibria including the special cases of niche Bertrand ($\alpha_1^* = \bar{\alpha}, \alpha_2^* = \bar{\alpha}$) and maximum differentiation ($\alpha_1^* = \bar{\alpha}, \alpha_2^* = \underline{\alpha}$). While consumer surplus is maximized under niche Bertrand, the most profitable corner pricing equilibrium is obtained under maximum differentiation. The following Proposition summarizes the equilibrium outcomes at the product design stage.

Proposition 2. *Define $\tilde{\alpha} = \min\left\{\left(\frac{\sqrt{2}}{g(\bar{s})\sqrt{\bar{s}}} + 2\sqrt{2}\sqrt{\bar{s}} - \sqrt{\underline{\alpha}}\right)^2, \underline{\alpha}\right\}$.*[20] *The equilibrium product design choices can be characterized as follows:*

- *If (8) holds: $\alpha_1^* = \bar{\alpha}$, $\alpha_2^* = \underline{\alpha}$. An interior pricing equilibrium is obtained.*

[20]Note that $\tilde{\alpha} \leq \bar{\alpha}$ if (8) does not hold.

- *If (8) does not hold:* $\alpha_1^* = \bar{\alpha}$, $\alpha_2^* \in [\underline{\alpha}, \bar{\alpha}]$ *such that* $\bar{\alpha} \in BR_1(\alpha_2^*)$. *A corner pricing equilibrium is obtained.* $\bar{\alpha} \in BR_1(\alpha_2^*)$ *if and only if:*
  - $\alpha_2^* \geq \tilde{\alpha}$ *or*
  - $(\bar{\alpha} - \alpha_2^*) \cdot \frac{1}{2} - (\sqrt{\bar{\alpha}} - \sqrt{\alpha_2^*}) \cdot \sqrt{2\bar{s}} \geq (\sqrt{\alpha_2^*} - \sqrt{\underline{\alpha}}) \cdot \frac{1}{\sqrt{2\hat{s}^*}} \cdot \frac{(1-G(\hat{s}^*))^2}{g(\hat{s}^*)}$, *where* $\hat{s}^*$ *is determined by equation (5) when evaluated at* α_2^* *and* $\underline{\alpha}$. *There exists* $\underset{\sim}{\alpha} > \underline{\alpha}$ *such that this condition is satisfied for* $\alpha_2^* \in [\underline{\alpha}, \underset{\sim}{\alpha})$.

Proof: See Appendix.

6 Conclusion

In this paper, I argue that product design can introduce vertical differentiation à la Gabszewicz and Thisse (1979) as well as Shaked and Sutton (1982). I study a duopoly model of directed search with multiproduct firms and show that it can be rewritten as a model of quality choice with single-product firms given the optimal search behavior of consumers. Product design resembles quality, as all consumers obtain a higher utility when visiting the firm with the more niche product design. Heterogeneous tastes for quality arise endogenously through heterogeneous search cost, where low search cost correspond to a high preference for quality.

Despite capturing the main mechanisms of classical models of vertical differentiation, my model is not formally equivalent to these textbook models. This is due to their assumption on the functional form of utilities. The utility of a consumer j when buying from firm k is assumed to equal $u_{jk} = r + \theta_j \cdot q_k$, where $r > 0$ denotes baseline utility, θ_j is a taste parameter and q_k denotes the quality of firm k. In the Appendix, I analyze such a model under the same technical assumptions than in my model. The equilibrium demand at the pricing stage turns out to be independent of the quality choices of firms. Whether a corner or an interior equilibrium arises and how demand is split in an interior equilibrium, solely depends on the distribution of consumers' tastes. The comparison with my model, which endogenously imposes a structure on the utility function through its search foundation, illustrates the restrictiveness of the assumption of a multiplicative structure of utility in models of vertical differentiation with full coverage.

A similar critique could be applied to my model: I rely on the functional form assumption of Larson (2013) to model product design. A more niche product design scales the variance of match values, but does not affect their mean. One might argue that the increased dispersion of match values comes at the cost of a lower average valuation. In that case, a more niche product design would not necessarily be preferred by all consumers. Instead, the preferred product design of a consumer could be a function of that consumer's search cost. Higher marginal costs for products with a more niche design could have a similar effect. While each consumer would still prefer a more niche product design at equal prices, it could be welfare-maximizing to allocate broader product designs to consumers with higher search cost. Lower average valuations and higher marginal costs for a more niche product design would thus shift the differentiation caused by product design from vertical to horizontal. As consumers would select into different product designs according to their search cost, differentiation through product design would still constitute a source of market power.

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Appendix

Consumer search under general beliefs

Even when consumers do not believe in symmetric pricing, each firm's products are ex-ante identical to them. As in section 3, consumers only visit the firm whose products have a higher reservation price.

Consider a consumer with search cost $s \in [\underline{s}, \bar{s}]$, which believes that firm k sets prices according to some CDF $L_k(.)$ with support $[\underline{p}, \bar{p}]$, where $\underline{p} \geq 0$.[21] Firm k's reservation price y_k is determined by the following equation:

$$\int_{\underline{z}}^{\bar{z}} \int_{\underline{p}}^{\bar{p}} \mathbb{1}\{\epsilon_\mu + \alpha_k \cdot z - p \geq y_k\} \cdot (\epsilon_\mu + \alpha_k \cdot z - p - y_k) dL_k(p) dF(z) = s$$

Consumers only search for products of the firm with the higher y_k. They stop if and only if the net utility of a product (weakly) exceeds y_k. Two immediate observations are useful for the following analysis. First, the reservation price y_k is strictly decreasing in s. Second, lowering the price of a particular product strictly increases the probability that a consumer buys that product. This follows from the fact that f is strictly positive and implies that consumers on average buy at a strictly lower price than the average price of firm k unless $L_k(.)$ is degenerate.

Furthermore, a useful lemma can be established:

Lemma 6. *Fix any product design $\alpha_k \in [\underline{\alpha}, \bar{\alpha}]$ and mean price $p_\mu > 0$ for firm k. Then, a consumer's reservation price y_k is higher under any non-degenerate price distribution than under symmetric pricing.*

Proof: The distribution of net utilities[22] under any form of price dispersion is a mean-preserving spread of the distribution of net utilities under symmetric pricing. It thus suffices to show that a consumer's incremental benefit function[23] would be strictly lower for any outside option y when facing a distribution of net match values H_1 than when facing a distribution of net match values H_2 which constitutes a mean preserving spread of H_1. Suppose that random variable X is distributed according to H_1 and that random variable Z is distributed according to H_2. Then, there is some random variable ϵ from a conditional

[21] The same argument holds for a CDF with support $[\underline{p}, \infty)$.

[22] The net utility of a product corresponds to its match value minus its price.

[23] As in Dogan and Hu (2022), I define the incremental benefit function as each consumer's incremental gain from one more search with a outside option of y at hand.

probability distribution $K(.|x)$ such that $Z = X + \epsilon$. Define $f(v) = \mathbb{1}\{v \geq y\} \cdot (v - y)$ and note that $f(.)$ is a convex function. For any $y \in \mathbb{R}$, Jensen's inequality can be used to show that:

$$\int_{-\infty}^{\infty} f(z) dH_2(z) = \int_{-\infty}^{\infty} \mathbb{E}[f(x+\epsilon)|x] dH_1(x) \geq \int_{-\infty}^{\infty} f(\mathbb{E}[x+\epsilon|x]) dH_1(x) = \int_{-\infty}^{\infty} f(x) dH_1(x)$$

Proof of Lemma 1

Suppose for a contradiction that there is an equilibrium candidate where at least one firm k sets different prices for its products according to a non-degenerate CDF L_k with mean $p_\mu > 0$ and support $[\underline{p}, \bar{p}]$, where $\underline{p} \geq 0$.

Suppose first that only one firm sets a non-degenerate price distribution and obtains no demand in equilibrium.[24] I index this firm with k and the other firm with $-k$. Then, there exists a search cost $s \in [\underline{s}, \bar{s}]$ such that a consumer with search cost s is indifferent between searching products of the two firms. Otherwise, firm $-k$ could profitably deviate by slightly increasing its price while still obtaining the entire demand.[25] Firm k could thus profitably deviate by setting an arbitrarily small, but strictly positive price p'_k, which is smaller than p_μ, for all of its products and capture a positive share of consumers.[26]

Suppose now that firm k sets a non-degenerate price distribution and obtains a positive share of consumers in equilibrium. Note from the discussion on consumer search under general beliefs that any consumer of firm k on average pays a price of less than p_μ and that a consumer's reservation price y_k is strictly decreasing in her search cost. Denote by $\underline{s}_k$ the infimum of the set of $s \in [\underline{s}, \bar{s}]$ such that consumers with search cost s buy at firm k in equilibrium. Denote by $\underline{y}_k$ the reservation price of firm k's products for a consumer with search cost $\underline{s}_k$. Define $\hat{p}$ as the lowest price at which a consumer with search cost $\underline{s}_k$ would buy according to her optimal rule of search when facing a product the highest possible match value of $\epsilon_\mu + \alpha_k \cdot \bar{\epsilon}$:

$$\hat{p} = \epsilon_\mu + \alpha_k \cdot \bar{\epsilon} - \underline{y}_k$$

Suppose first that $\hat{p} \geq p_\mu$. Then, firm k could profitably deviate to symmetric pricing at

[24] This implies that the other firm employs symmetric pricing and obtains the entire demand.

[25] This hold true independent of consumers' off-path beliefs about price dispersion at firm $-k$. If consumers believed that firm $-k$ would set a non-degenerate price distribution when observing an off-path average price, this would not lead to a lower reservation price of firm $-k$ than under symmetric pricing due to Lemma 6.

[26] Consumers' reservation price is minimized when they believe in symmetric pricing at p'_k. The deviation price p'_k can however be chosen low enough such that consumers are better off – even under symmetric pricing – than under any non-degenerate price distribution $L_k(.)$ with mean $p_\mu > 0$.

p_μ. As the average price remains constant, demand would not be affected. Suppose now that $\hat{p} < p_\mu$ and that there is no mass on prices strictly below $\hat{p}$, i.e. that $\lim_{p \to \hat{p}^-} L_k(p) = 0$. Then, the expected gains from an additional search when facing a product with a match value of $\epsilon_\mu + \alpha_k \cdot \bar{\epsilon}$ and a price of $\hat{p}$ equal zero. This implies that $\underline{y}_k < \epsilon_\mu + \alpha_k \cdot \bar{\epsilon} - \hat{p}$ which contradicts the definition of $\hat{p}$. Lastly, suppose that $\hat{p} < p_\mu$ and that there is a positive mass on prices strictly below $\hat{p}$, i.e. that $\lim_{p \to \hat{p}^-} L_k(p) > 0$. Then, define distribution $L'_k(.)$ as follows:

$$L'_k(p) = \begin{cases} 0, & \text{if } p \in [0, \hat{p}), \\ L_k(p), & \text{if } p \geq \hat{p}, \end{cases}$$

Under $L'_k(.)$, all mass below $\hat{p}$ is shifted to a mass point at $\hat{p}$. Denote by $p'_\mu > p_\mu$ the mean of that distribution. Define $\tilde{p}$ such that:

$$\int_{\tilde{p}}^{\infty} (p - \tilde{p}) dL_k(p) = p'_\mu - p_\mu$$

Clearly $\tilde{p} > p_\mu$, as $\hat{p} < p_\mu$. Define $L''_k(.)$ as follows:

$$L''_k(p) = \begin{cases} 0, & \text{if } p \in [0, \hat{p}), \\ L_k(p), & \text{if } p \in [\hat{p}, \tilde{p}), \\ 1, & \text{if } p \geq \tilde{p}, \end{cases}$$

Under $L''_k(.)$, the entire mass above $\tilde{p}$ is shifted to a mass point on $\tilde{p}$ in order to compensate for the increase in average price through $L'_k(.)$. It is straightforward to see that the mean of $L''_k(.)$ equals p_μ. Firm k could profitably deviate to setting prices according to $L''_k(.)$. As the average price remains constant, consumers would not adjust their search behavior. They would however buy at a higher average price under $L''_k(.)$. Switching from $L_k(.)$ to $L'_k(.)$ would increase the average purchase price conditional on consumer's equilibrium rule of search by more than $p'_\mu - p_\mu$.[27] This is due to the fact that consumers are more likely to purchase at lower prices. By the same reasoning, switching from $L'_k(.)$ to $L''_k(.)$ would decrease the average purchase price conditional on consumer's equilibrium rule of search by less than $p'_\mu - p_\mu$.

[27] There are three arguments to consider here. First, the probability to obtain a purchase price smaller or equal than $\hat{p}$ under $L_k(.)$ is higher than $L_k(\hat{p})$. Second, the expected purchase price conditional on obtaining a purchase price smaller or equal than $\hat{p}$ increases by more than $\frac{p'_\mu - p_\mu}{L_k(\hat{p})}$ when switching to $L'_k(.)$. Third, the probability to obtain a higher purchase price than $\hat{p}$ is higher under $L'_k(.)$.

Proof of Lemma 3

Suppose that firm 2 sets price $p_2 \geq 0$. If firm 1 sets $p_1 \in [NCS_1(\bar{s}) - NCS_2(\bar{s}) + p_2, NCS_1(\underline{s}) - NCS_2(\underline{s}) + p_2]$, condition (3) holds. Suppose that $p_1 < NCS_1(\bar{s}) - NCS_2(\bar{s}) + p_2$. Then firm 1 could deviate to $p_1 = NCS_1(\bar{s}) - NCS_2(\bar{s}) + p_2$ and sell the same quantity at a higher price. Suppose that $p_1 > NCS_1(\underline{s}) - NCS_2(\underline{s}) + p_2$. Then firm 1 obtains zero profits, which is the lowest possible profit and is thus weakly better off to set any other price.[28] The symmetric argument applies to firm 2.

Proof of Proposition 1

In the main body of the paper, I have already shown that there is a unique equilibrium candidate for each subgame. I now derive a sufficient condition under which this candidate actually constitutes an equilibrium. For this purpose, I establish log-concavity of the profit functions.

Suppose first that (6) does not hold so that the unique equilibrium candidate constitutes a corner equilibrium. Clearly, it is optimal for firm 2 to set $p_2 = 0$. Any higher price would also generate zero demand and lead to zero profits. Given firm 2's price, firm 1 solves the optimization problem given by (7). Taking logs and differentiating twice with respect to s yields:

$$\underbrace{\frac{1}{\left(\left(\sqrt{\alpha_1}+\sqrt{\alpha_2}\right)\cdot\frac{1}{\sqrt{2}}\cdot\sqrt{s}-2s\right)^2}}_{>0} \cdot \left(\left(\sqrt{\alpha_1}+\sqrt{\alpha_2}\right)\cdot\frac{1}{2^{1.5}\cdot\sqrt{s}}-2\right) + \underbrace{\frac{g'(s)\cdot G(s)-g(s)^2}{G(s)^2}}_{\leq 0 \text{ due to log-concavity of } G}$$

Thus, the profit function is strictly log-concave if $\left(\sqrt{\alpha_1}+\sqrt{\alpha_2}\right)\cdot\frac{1}{2^{1.5}\cdot\sqrt{s}}-2<0$ for all $s \in [\underline{s}, \bar{s}]$. This holds true for all subgames if $\bar{\alpha} < 8 \cdot \underline{s}$.

Suppose now that (6) does hold so that the unique equilibrium candidate constitutes an interior equilibrium. Given p_2^*, firm 1 solves the following problem:

$$\max_{\hat{s}\in[\underline{s},\bar{s}]} \left(p_2^* + NCS_1(\hat{s}) - NCS_2(\hat{s})\right)\cdot G(\hat{s})$$

[28] For firm 1, it is straightforward to show a strictly profitable deviation here. However, this argument would not apply to firm 2.

Taking logs, differentiating twice and plugging in for p_2^* yields:

$$\underbrace{\frac{1}{\frac{1}{\sqrt{2\hat{s}^*}}\cdot\frac{1-G(\hat{s}^*)}{g(\hat{s}^*)}+(\sqrt{\alpha_1}+\sqrt{\alpha_2})\cdot\frac{1}{2}-\sqrt{2s}}\cdot\frac{1}{2s}}_{>0 \text{ as } \bar{s}\leq\frac{1}{2}\cdot\underline{\alpha} \text{ by assumption}}$$

$$\cdot\left(\frac{1}{\sqrt{2s}}-\frac{1}{\frac{1}{\sqrt{2\hat{s}^*}}\cdot\frac{1-G(\hat{s}^*)}{g(\hat{s}^*)}+(\sqrt{\alpha_1}+\sqrt{\alpha_2})\cdot\frac{1}{2}-\sqrt{2s}}\right)$$

$$+\quad\underbrace{\frac{g'(s)G(s)-g(s)^2}{G(s)^2}}_{\leq 0 \text{ due to log-concavity of } G}$$

Thus, the profit function is strictly log-concave if $\frac{1}{\sqrt{2s}}-\frac{1}{\frac{1}{\sqrt{2\hat{s}^*}}\cdot\frac{1-G(\hat{s}^*)}{g(\hat{s}^*)}+(\sqrt{\alpha_1}+\sqrt{\alpha_2})\cdot\frac{1}{2}-\sqrt{2s}}<0$ for all $s\in[\underline{s},\bar{s}]$ and $\hat{s}^*\in[\text{med}(G),\bar{s}]$. This holds true for all subgames if $\sqrt{\bar{\alpha}}<2\cdot\sqrt{2\underline{s}}-\frac{\sqrt{2}}{\text{med}(G)}\cdot\frac{1}{4\cdot g(\text{med}(G))}$. Note that this condition implies that $\bar{\alpha}<8\cdot\underline{s}$ and that $\bar{\alpha}<8\cdot\text{med(G)}$.

For firm 2, consider the pricing representation of the profit maximization problem. Given p_1^*, firm 2 solves:

$$\max_{p_2\geq 0} p_2\cdot\big(1-G(\hat{s}(p_1^*,p_2))\big)$$

Lemma 3 shows that firm 2 is (weakly) better off setting a price such that $NCS_1(\hat{s})-NCS_2(\hat{s})=p_1-p_2$. Using that restriction, I can explicitly solve this equation for $\hat{s}$: $\hat{s}=\frac{1}{2}\cdot\left((\sqrt{\alpha_1}+\sqrt{\alpha_2})\cdot\frac{1}{2}-\frac{p_1^*-p_2}{\sqrt{\alpha_1}-\sqrt{\alpha_2}}\right)^2$. It turns out that $\hat{s}$ is strictly increasing and strictly convex in p_2:

$$\frac{\partial\hat{s}}{\partial p_2}=\left((\sqrt{\alpha_1}+\sqrt{\alpha_2})\cdot\frac{1}{2}-\frac{p_1^*-p_2}{\sqrt{\alpha_1}-\sqrt{\alpha_2}}\right)\cdot\frac{1}{\sqrt{\alpha_1}-\sqrt{\alpha_2}}>0$$

$$\frac{\partial^2\hat{s}}{\partial p_2^2}=\frac{1}{(\sqrt{\alpha_1}-\sqrt{\alpha_2})^2}>0$$

The SOC of the logarithm of the profit function of firm 2 takes the following form:

$$-\frac{1}{p_2^2}-\frac{g(\hat{s})}{1-G(\hat{s})}\cdot\frac{\partial^2\hat{s}}{\partial p_2^2}-\underbrace{\frac{g'(\hat{s})\big(1-G(\hat{s})\big)+g(\hat{s})^2}{\big(1-G(\hat{s})\big)^2}}_{>0 \text{ due to log-concavity of } 1-G}\cdot\left(\frac{\partial\hat{s}}{p_2}\right)^2<0$$

Thus, the profit function is strictly log-concave.

Proof of Lemma 4

The comparative statics of $\hat{s}^*$ with respect to α_1 and α_2 are useful for the proof. Applying the implicit function theorem to (5) yields:

$$\frac{\partial \hat{s}^*}{\partial \alpha_k} = \frac{\frac{1}{2^{1.5}} \cdot \frac{1}{\sqrt{\alpha_k}} \cdot \sqrt{\hat{s}^*}}{\frac{2 \cdot g(\hat{s}^*)^2 - (2 \cdot G(\hat{s}^*) - 1) \cdot g'(\hat{s}^*)}{g(\hat{s}^*)^2} + 2 - \frac{\sqrt{\alpha_1} + \sqrt{\alpha_2}}{2^{1.5} \cdot \sqrt{\hat{s}^*}}} > 0 \text{ for } k \in \{1,2\}$$

where the denominator is greater than zero due to the restriction on $\bar{\alpha}$ and the log-concavity of G and $1-G$. The equilibrium threshold search cost is increasing in α_1 and α_2.

It follows that the interior equilibrium profits of firm 2 ($\pi_2^i(\alpha_1, \alpha_2)$) are strictly decreasing in α_2 (for $\alpha_2 < \alpha_1$):

$$\begin{aligned}
\frac{\partial \pi_2^i}{\partial \alpha_2} &= \frac{\partial \left(\sqrt{\alpha_1} - \sqrt{\alpha_2}\right) \cdot \frac{1}{\sqrt{2\hat{s}^*}} \cdot \frac{\left(1 - G(\hat{s}^*)\right)^2}{g(\hat{s}^*)}}{\partial \alpha_2} \\
&= -\frac{1}{2\sqrt{\alpha_2}} \cdot \frac{1}{\sqrt{2\hat{s}^*}} \cdot \frac{\left(1 - G(\hat{s}^*)\right)^2}{g(\hat{s}^*)} \\
&\quad - \left(\sqrt{\alpha_1} - \sqrt{\alpha_2}\right) \cdot \frac{1}{(2\hat{s}^*)^{1.5}} \cdot \frac{\partial \hat{s}^*}{\partial \alpha_2} \cdot \frac{\left(1 - G(\hat{s}^*)\right)^2}{g(\hat{s}^*)} \\
&\quad - \left(\sqrt{\alpha_1} - \sqrt{\alpha_2}\right) \cdot \frac{1}{\sqrt{2\hat{s}^*}} \cdot \underbrace{\frac{2 \cdot \left(1 - G(\hat{s}^*)\right) \cdot g(\hat{s}^*)^2 + g'(\hat{s}^*) \cdot \left(1 - G(\hat{s}^*)\right)^2}{g(\hat{s}^*)^2}}_{\geq 0 \text{ due to log-concavity of } 1-G} \cdot \frac{\partial \hat{s}^*}{\partial \alpha_2} \\
&< 0
\end{aligned}$$

I now turn to the interior equilibrium profits of firm 1 ($\pi_1^i(\alpha_1, \alpha_2)$). Taking the first derivative of the logarithmized profits yields:

$$\begin{aligned}
\frac{\partial \log \pi_1^i}{\partial \alpha_1} &= \frac{\partial \ \log\left(\sqrt{\alpha_1} - \sqrt{\alpha_2}\right) - \log\left(\sqrt{2\hat{s}^*}\right) + \log\left(\frac{G(\hat{s}^*)^2}{g(\hat{s}^*)}\right)}{\partial \alpha_1} \\
&= \frac{1}{\sqrt{\alpha_1} - \sqrt{\alpha_2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{\alpha_1}} - \frac{1}{2\hat{s}^*} \cdot \frac{\partial \hat{s}^*}{\partial \alpha_1} + \underbrace{\frac{2g(\hat{s}^*)^2 - g'(\hat{s}^*) \cdot G(\hat{s}^*)}{g(\hat{s}^*) \cdot G(\hat{s}^*)}}_{\geq 0 \text{ due to log-concavity of } G} \cdot \frac{\partial \hat{s}^*}{\partial \alpha_1}
\end{aligned}$$

A sufficient condition for strictly increasing profits is given by:

$$\frac{1}{\sqrt{\alpha_1} - \sqrt{\alpha_2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{\alpha_1}} > \frac{1}{2\hat{s}^*} \cdot \frac{\partial \hat{s}^*}{\partial \alpha_1} \tag{9}$$

It follows from log-concavity of G and $1-G$ that:

$$\frac{\partial \hat{s}^*}{\partial \alpha_1} = \frac{\frac{1}{2^{1.5}} \cdot \frac{1}{\sqrt{\alpha_1}} \cdot \sqrt{\hat{s}^*}}{\frac{2 \cdot g(\hat{s}^*)^2 - (2 \cdot G(\hat{s}^*)-1) \cdot g'(\hat{s}^*)}{g(\hat{s}^*)^2} + 2 - \frac{\sqrt{\alpha_1}+\sqrt{\alpha_2}}{2^{1.5} \cdot \sqrt{\hat{s}^*}}} \leq \frac{\frac{1}{2^{1.5}} \cdot \frac{1}{\sqrt{\alpha_1}} \cdot \sqrt{\hat{s}^*}}{2 - \frac{\sqrt{\alpha_1}+\sqrt{\alpha_2}}{2^{1.5} \cdot \sqrt{\hat{s}^*}}}$$

A sufficient condition for (9) is thus:

$$\frac{1}{\sqrt{\alpha_1} - \sqrt{\alpha_2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{\alpha_1}} > \frac{1}{2\hat{s}^*} \cdot \frac{\frac{1}{2^{1.5}} \cdot \frac{1}{\sqrt{\alpha_1}} \cdot \sqrt{\hat{s}^*}}{2 - \frac{\sqrt{\alpha_1}+\sqrt{\alpha_2}}{2^{1.5} \cdot \sqrt{\hat{s}^*}}}$$

This can be simplified to:

$$2\sqrt{2\hat{s}^*} > \sqrt{\alpha_1}$$

As $\hat{s}^* \geq \text{med}(G)$ and $\alpha_1 \leq \bar{\alpha}$, this is implied by $\sqrt{\bar{\alpha}} < 2 \cdot \sqrt{2\underline{s}} - \frac{\sqrt{2}}{\text{med}(G)} \cdot \frac{1}{4 \cdot g(\text{med}(G))}$.

In any equilibrium that leads to an interior pricing equilibrium and does not satisfy the conditions of the Lemma, there would thus be a profitable deviation for one of the two firms.

Proof of Lemma 5

Consider any equilibrium candidate (α_1, α_2) such that $\max(\alpha_1, \alpha_2) \neq \bar{\alpha}$. If $\alpha_1 = \alpha_2$, both firms make zero profits and could profitably deviate to $\bar{\alpha}$. Suppose from now on that $\alpha_1 \neq \alpha_2$. First, consider (α_1, α_2) such that condition (6) is not satisfied. Then, there is a corner pricing equilibrium and one firm makes zero profits. This firm could profitably deviate to $\bar{\alpha}$ and get the entire demand at a strictly positive price. Now consider (α_1, α_2) such that (6) is satisfied. There would thus be an interior pricing equilibrium. Suppose without loss of generality that $\alpha_1 > \alpha_2$. Then there exists $\alpha' > \alpha_1$ such that (α', α_2) satisfy (6). According to Lemma 4, firm 1 should profitably deviate to α'.

Proof of Proposition 2

Suppose that condition (8) holds. Then, the equilibrium characterization follows directly from Lemma 4. Suppose now that condition (8) does not hold. By Lemma 5, firm 1 sets $\bar{\alpha}$ in any equilibrium. Any $\alpha_2 \in [\underline{\alpha}, \bar{\alpha}]$ yields zero profits and constitutes a best response to $\bar{\alpha}$. It thus remains to be shown that $\bar{\alpha}$ constitutes a best response to $\alpha_2 \in [\underline{\alpha}, \bar{\alpha}]$ under the conditions of Proposition 2.

Fix any $\alpha_2 \in [\underline{\alpha}, \bar{\alpha}]$. First, consider $\alpha_2 \geq \tilde{\alpha}$. This implies that any $\alpha_1 \in [\underline{\alpha}, \bar{\alpha}]$ leads to a corner pricing equilibrium. Since $\alpha_1 \leq \alpha_2$ yields zero profits for firm 1 and firm 1's corner pricing equilibrium profits are strictly increasing in α_1 for $\alpha_1 > \alpha_2$, setting $\bar{\alpha}$ constitutes the unique best response for firm 1.

Consider now $\alpha_2 < \tilde{\alpha}$. Define $\hat{\alpha}_1$ such that, given α_2, condition (6) does not hold true when $\alpha_1 \geq \hat{\alpha}_1$, that is:

$$\hat{\alpha}_1 = \left(\frac{\sqrt{2}}{g(\bar{s}) \cdot \sqrt{\bar{s}}} + 2 \cdot \sqrt{2} \cdot \sqrt{\bar{s}} - \sqrt{\alpha_2}\right)^2$$

Suppose first that $\hat{\alpha}_1 > \alpha_2$. Clearly, firm 1's corner pricing equilibrium profits are strictly increasing in α_1 for $\alpha_1 \geq \hat{\alpha}_1$. Two possibilities to deviate remain to be checked: First, firm 1 could deviate to $\alpha_1' \in (\alpha_2, \hat{\alpha_1})$, which leads to an interior pricing equilibrium with a competitive advantage for firm 1. Such deviations are not profitable according to the following Lemma.

Lemma 7. *Fix $\alpha_2 \in [\underline{\alpha}, \bar{\alpha})$ and suppose that condition (8) does not hold. Then, firm 1 is weakly better off to choose $\bar{\alpha}$ than any $\alpha_1 \in [\alpha_2, \bar{\alpha}]$.*

Proof: By Lemma 4, firm 1's interior pricing equilibrium profits are strictly increasing in α_1 for $\alpha_1 \in (\alpha_2, \hat{\alpha}_1)$. Furthermore, firm 1's corner pricing equilibrium profits are strictly increasing in α_1 for $\alpha_1 \in [\hat{\alpha}_1, \bar{\alpha}]$. It thus remains to be shown that the corner pricing equilibrium profits at $\bar{\alpha}$ are higher than the limit of interior pricing equilibrium profits when $\alpha_1 \to \hat{\alpha}_1$. The corner pricing equilibrium profits at $\bar{\alpha}$ are equal to:

$$(\bar{\alpha} - \alpha_2) \cdot \frac{1}{2} - (\sqrt{\bar{\alpha}} - \sqrt{\alpha_2}) \cdot \sqrt{2\bar{s}}$$

An upper bound for the interior equilibrium pricing profits can be derived as follows:[29]

$$\begin{aligned}
&(\sqrt{\alpha_1} - \sqrt{\alpha_2}) \cdot \frac{1}{\sqrt{2\hat{s}^*}} \cdot \frac{G(\hat{s}^*)^2}{g(\hat{s}^*)} \\
&\xrightarrow{\alpha_1 \to \hat{\alpha}_1} (\sqrt{\hat{\alpha}_1} - \sqrt{\alpha_2}) \cdot \frac{1}{\sqrt{2\bar{s}}} \cdot \frac{1}{g(\bar{s})} \\
&\leq (\sqrt{\bar{\alpha}} - \sqrt{\alpha_2}) \cdot \frac{1}{\sqrt{2\bar{s}}} \cdot \frac{1}{g(\bar{s})}
\end{aligned}$$

Corner pricing equilibrium profits lie above that upper bar if and only if the following condition is satisfied:

$$\sqrt{\bar{\alpha}} + \sqrt{\alpha_2} \geq 2\sqrt{2\bar{s}} + \frac{\sqrt{2}}{\sqrt{\bar{s}} \cdot g(\bar{s})}$$

[29] Again, $\hat{s}^*$ is determined by equation (5).

This is implied by the fact that condition (8) does not hold, which completes the proof of the Lemma.

Second, firm 1 could deviate to $\alpha_1' \in [\underline{\alpha}, \alpha_2]$. By Lemma 4, the best deviation in this interval is to set $\underline{\alpha}$. This deviation is not profitable if and only if

$$(\bar{\alpha}-\alpha_2)\cdot\frac{1}{2}-(\sqrt{\bar{\alpha}}-\sqrt{\alpha_2})\cdot\sqrt{2\bar{s}} \geq (\sqrt{\alpha_2}-\sqrt{\underline{\alpha}})\cdot\frac{1}{\sqrt{2\hat{s}^*}}\cdot\frac{(1-G(\hat{s}^*))^2}{g(\hat{s}^*)}$$

where $\hat{s}^*$ is determined by equation (5) when evaluated at α_2 and $\underline{\alpha}$. Clearly, the right-hand-side of this equation converges to zero as $\alpha_2 \to \underline{\alpha}$, while the left-hand-side converges to a strictly positive value. Thus, there exists $\underset{\sim}{\alpha} > \underline{\alpha}$ such that this condition is satisfied for $\alpha_2 \in [\underline{\alpha}, \underset{\sim}{\alpha})$.

Suppose now that $\hat{\alpha}_1 \leq \alpha_2$. Then, deviations to $\alpha_1' \in [\hat{\alpha}_1, \alpha_2]$ yield zero profits and thus only deviations to $\alpha_1' \in [\underline{\alpha}, \hat{\alpha}_1)$ need to be considered. By the previous arguments, the same condition as under $\hat{\alpha}_1 > \alpha_2$ is obtained.

Comparison with textbook models of vertical differentiation

In this subsection, I solve a standard model of vertical differentiation similar to Belleflamme and Peitz (2015). There are two single-product firms and a continuum of consumers. The utility of consumer j when buying from firm k is equal to $u_{jk} = r + \theta_j \cdot q_k$. Baseline utility $r > 0$ is assumed to be sufficiently large such that all consumers buy a product (full market coverage). Taste parameter θ_j is drawn independently from a differentiable CDF H with support $[\underline{\theta}, \bar{\theta}]$. The same technical assumptions as on G are imposed, i.e. h is assumed to be strictly positive and H as well as $1-H$ are assumed to be log-concave. The quality of a firm's product is represented by q_k. Firms face zero marginal cost and play a two-stage game. They first simultaneously choose their qualities from an interval of $[\underline{q}, \bar{q}]$.[30] Secondly, they compete in prices.

Pricing: If both firms set the same quality, they would price at marginal cost. In the following, I assume WLOG that $q_1 > q_2$. It is straightforward to calculate the threshold taste parameter $\hat{\theta}$:[31]

$$\hat{\theta} = \frac{p_1 - p_2}{q_1 - q_2}$$

While consumers with taste parameter $\theta < \hat{\theta}$ will buy from firm 2, consumers with a search

[30]There is no cost associated with quality.

[31]I use the equivalent of Lemma 3 here without explicitly stating it.

cost parameter of $\theta > \hat{\theta}$ will buy from firm 1. Using that, the profit functions of the two firms can be derived:

$$\pi_1(p_1,p_2) = p_1 \cdot \left(1 - H\left(\frac{p_1 - p_2}{q_1 - q_2}\right)\right) \qquad \pi_2(p_1,p_2) = p_2 \cdot H\left(\frac{p_1 - p_2}{q_1 - q_2}\right)$$

As in my model, the profit maximization problem could also be expressed in terms of choosing a threshold taste parameter:

$$\text{Firm 1:} \max_{\hat{\theta} \in [\underline{\theta}, \bar{\theta}]} \left(p_2 + \hat{\theta} \cdot (q_1 - q_2)\right) \cdot \left(1 - H(\hat{\theta})\right)$$
$$\text{Firm 2:} \max_{\hat{\theta} \in [\underline{\theta}, \bar{\theta}]} \left(p_1 - \hat{\theta} \cdot (q_1 - q_2)\right) \cdot H(\hat{\theta})$$

Taking FOCs yields the following equations:

$$p_1^* = \frac{1 - H(\hat{\theta}^*)}{h(\hat{\theta}^*)} \cdot (q_1 - q_2)$$

$$p_2^* = \frac{H(\hat{\theta}^*)}{h(\hat{\theta}^*)} \cdot (q_1 - q_2)$$

It follows from the equivalent of Lemma 3, that the following condition needs to be fulfilled in any equilibrium:

$$p_1^* - p_2^* = \hat{\theta}^* \cdot (q_1 - q_2)$$

Using the FOCs, this boils down to the following equation:

$$\frac{1 - 2H(\hat{\theta}^*)}{h(\hat{\theta}^*)} = \hat{\theta}^* \tag{10}$$

While the LHS of this equation is strictly decreasing due to the log-concavity of H and $1 - H$, the RHS is strictly increasing. There is thus at most one interior pricing equilibrium candidate, which cannot be located in $[med(H), \bar{\theta}]$ as the LHS is negative on that interval. Such a candidate exists if and only if the following inequality holds true:

$$1 > \underline{\theta} \cdot h(\underline{\theta}) \tag{11}$$

The negation of (11) constitutes a necessary condition for a corner pricing equilibrium. The FOC of firm 1's profits given $p_2 = 0$ (using the $\hat{\theta}$-representation) evaluated at $\underline{\theta}$ is negative if and only if $1 \leq \underline{\theta} \cdot h(\underline{\theta})$. It remains to establish the log-concavity of profit functions. The second derivatives of the log-profits of firm 1 and 2 (using the p-representation) are given by the following two equations:

$$\text{Firm 1: } -\frac{1}{(p_1)^2}-\frac{h'(\hat{\theta})\cdot\left(1-H(\hat{\theta})\right)+h(\hat{\theta})^2}{\left(1-H(\hat{\theta})\right)^2}\cdot\left(\frac{1}{q_1-q_2}\right)^2<0$$
$$\text{Firm 2: } -\frac{1}{(p_2)^2}+\frac{h'(\hat{\theta})\cdot H(\hat{\theta})-h(\hat{\theta})^2}{H(\hat{\theta})^2}\cdot\left(\frac{1}{q_1-q_2}\right)^2<0$$

The inequalities follow from the fact that H and $1-H$ are log-concave.

Thus, there exists a unique pure-strategy equilibrium at the pricing stage that can be characterized as follows:

- If (11) holds, there is an interior equilibrium: $\hat{\theta}^* \in [\underline{\theta}, med(H)]$ is implicitly defined by (10), $p_1^* = \frac{1-H(\hat{\theta}^*)}{h(\hat{\theta}^*)}\cdot(q_1-q_2)$, $p_2^* = \frac{H(\hat{\theta}^*)}{h(\hat{\theta}^*)}\cdot(q_1-q_2)$.
- If (11) does not hold, there is a corner equilibrium: $\hat{\theta}^* = \underline{\theta}$, $p_1^* = \underline{\theta}\cdot(q_1-q_2)$, $p_2^* = 0$.

Product design: There is a crucial observation from comparative statics of the equilibrium at the pricing stage: Neither (10) nor (11) depend on q_1 or q_2. The quality choices of firms do thus not affect equilibrium demand at the pricing stage. This implies that equilibrium profits in an interior pricing equilibrium are strictly increasing in (q_1-q_2) for both firms. If (11) holds, firms thus choose maximum differentiation. If (11) does not hold, there is a corner pricing equilibrium regardless of firms' quality choices. Setting $\bar{q}$ is then a weakly dominant strategy. In equilibrium, one firm sets $\bar{q}$ and the other firm sets any $q \in [\underline{q}, \bar{q}]$.[32] The equilibrium product design choices can be characterized as follows:

- If (11) holds: $q_1^* = \bar{q}$, $q_2^* = \underline{q}$. An interior pricing equilibrium is obtained.
- If (11) does not hold: $q_1^* = \bar{q}$, $q_2^* \in [\underline{q}, \bar{q}]$. A corner pricing equilibrium is obtained.

[32] Any $q \in [\underline{q}, \bar{q}]$ yields zero profits and constitutes a best response to $\bar{q}$.